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THE TORSION OF A GROWING CYLINDER BY A RIGID STAMP*

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The contact problem of the torsion of a viscoelastic ageing, growing cylinder by a rigid stamp is considered. Dual series equations reflecting the mathematical content of the problem of different stages of the growing process are derived and investigated. The results of a numerical analysis and the singularities of the qualitative behaviour of the fundamental characteristics are discussed.

1. Formulation and derivation of dual series equations of the contact problem. We will assume that a fairly long circular cylinder of length l and radius b_0 (the ratio of l to b_0 is fairly large) is fabricated from an ageing viscoelastic material at zero time. One of the cylinder endfaces is clamped at a non-deformable base while a rigid stamp of circular planform with a flat bottom of radius $a < b_0$ is coupled coaxially to the other. At a time τ_0 a torque $M(t)$ starts to act on the stamp, rotating it through an angle $\alpha(t)$. The cylinder side surface is stress-free.

At a time τ_1 substance influx to the cylinder side surface starts. The new incremental elements are not stressed and the time of their fabrication coincides with the time of initial body fabrication. Adhesion of each element with non-deformable base from the clamped endface side occurs at the time of attachment.

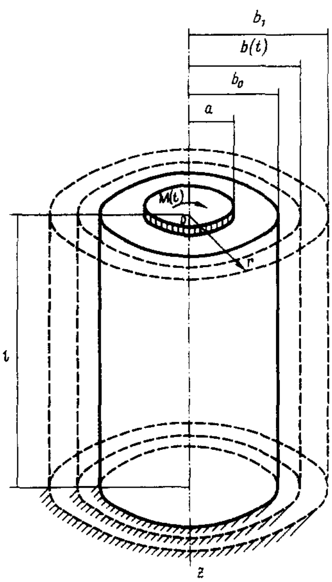

Fig.1

The law of cylinder growth is given completely by the function $b(t)$ that characterizes the change in its radius with time. Naturally $b(\tau_1) = b_0$.

The growing ceases at a time τ_2. At that time the cylinder radius takes the value b_1 ($b(\tau_2) = b_1$), and its side surface is free of any action even at $t \geqslant \tau_2$. The contact growing problem is studied within the framework of a quasi-static approximation in the absence of body forces /1, 2/ (Fig.1).

The cylinder is considered to be relatively long during the growth process and after its cessation (the ratios $l/b(t)$ and l/b_1 are fairly large).

Consider the fundamental relations of the problem in the time interval $t \in [\tau_0, \tau_1]$. We have for the initial visco-elastic ageing cylinder

$$\frac{\partial \tau_{r\varphi}}{\partial r} + \frac{\partial \tau_{\varphi z}}{\partial z} + \frac{2\tau_{r\varphi}}{r} = 0 \quad (\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}) \tag{1.1}$$

$$z = 0, \quad 0 \leqslant r \leqslant a: \; u_\varphi = \alpha(t) r; \quad z = 0, \quad a \leqslant r \leqslant b_0: \; \tau_{\varphi z} = 0$$

$$r = b_0, \quad 0 \leqslant z \leqslant l: \; \tau_{r\varphi} = 0; \quad z = l, \quad 0 \leqslant r \leqslant b_0: \; u_\varphi = 0$$

$$\varepsilon_{r\varphi} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right), \quad \varepsilon_{\varphi z} = \frac{1}{2}\frac{\partial u_\varphi}{\partial z} \left(\boldsymbol{\varepsilon} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}]\right)$$

$$\boldsymbol{\sigma} = 2G(t)(\mathbf{I} + \mathbf{F}(\tau_0, t))\boldsymbol{\varepsilon}, \quad (\mathbf{I} - \mathrm{S}(\tau_0, t)) = (\mathbf{I} + \mathbf{F}(\tau_0, t))^{-1}$$

$$\mathrm{S}(\tau_0, t) f(t) = \int_{\tau_0}^{t} f(\tau) K_1(t, \tau)\, d\tau, \quad K_1(t, \tau) = G(\tau) \frac{\partial}{\partial \tau}\left[\frac{1}{G(\tau)} + \omega(t, \tau)\right]$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and strain tensors with non-zero components $\tau_{r\varphi}, \tau_{\varphi z}$ and $\varepsilon_{r\varphi}, \varepsilon_{\varphi z}$ respectively, $\mathbf{u}$ is the displacement vector with a single non-zero component u_φ, $K_1(t, \tau)$, $\omega(t, \tau)$ and $G(t)$ is the creep kernel, the measure of creep, and the modulus of elastically-instantaneous deformation under pure shear. The arguments r, z, t in cases when this does not complicate reading the formulas will be omitted.

We set

$$\boldsymbol{\sigma}^\circ = (\mathbf{I} - \mathrm{S}(\tau_0, t))\boldsymbol{\sigma} G^{-1} \tag{1.2}$$

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and we act on the expression from (1.1) containing σ and its components with the operator $(\mathbf{I} - \mathbf{S}(\tau_0, t))$. Then taking (1.2) into account, we obtain the following boundary-value problem

$$\frac{\partial \tau_{r\varphi}^\circ}{\partial r} + \frac{\partial \tau_{\varphi z}^\circ}{\partial z} + \frac{2\tau_{r\varphi}^\circ}{r} = 0 \quad (\nabla \cdot \sigma^\circ = \mathbf{0}) \tag{1.3}$$
$$z = 0,\ 0 \leqslant r \leqslant a:\ u_\varphi = \alpha(t) r;\ z = 0,\ a \leqslant r \leqslant b_0:\ \tau_{\varphi z}^\circ = 0$$
$$r = b_0,\ 0 \leqslant z \leqslant l:\ \tau_{r\varphi}^\circ = 0;\ z = l,\ 0 \leqslant r \leqslant b_0:\ u_\varphi = 0$$
$$\varepsilon = {}^1/_2 [\nabla \mathbf{u} + (\nabla \mathbf{u})^T], \quad \sigma^\circ = 2\varepsilon$$

On the basis of (1.3) we determine that the displacement u_φ satisfies the equation

$$\mathbf{D} u_\varphi = \frac{\partial^2 u_\varphi}{\partial r^2} + \frac{\partial^2 u_\varphi}{\partial z^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r^2} = 0 \tag{1.4}$$

Following /3/ we take the solution of (1.4) in the form (see /4-7/)

$$u_\varphi(r, z, t) = l b_0^{-1} d_0(t)\, r\,(1 - z l^{-1}) + \tag{1.5}$$
$$\sum \delta_n^{-1} d_n(t)\, J_1(r\delta_n)\, \operatorname{sh}[\delta_n(l - z)]\, \operatorname{sh}^{-1}(\delta_n l)$$

where $d_k(t)$ $(k = 0, \ldots, \infty)$ are unknown functions of time, δ_n $(n = 1, \ldots, \infty)$ are undetermined constants, and $J_\nu(x)$ is the Bessel function of order ν. Here and henceforth the summation is from $n = 1$ to $n = \infty$.

We note that expression (1.5) for the displacement u_φ satisfies the boundary condition from (1.3) on the clamped endface of the cylinder for $z = l$ and enables us to write the tensor components of the operator stresses σ° in the form (see (1.1) and (1.3))

$$\tau_{\varphi z}^\circ(r, z, t) = -b_0^{-1} d_0(t)\, r - \tag{1.6}$$
$$\sum d_n(t)\, J_1(r\delta_n)\, \operatorname{ch}[\delta_n(l - z)]\, \operatorname{sh}^{-1}(\delta_n l)$$
$$\tau_{r\varphi}^\circ(r, z, t) = -\sum d_n(t)\, J_2(r\delta_n)\, \operatorname{sh}[\delta_n(l - z)]\, \operatorname{sh}^{-1}(\delta_n l)$$

Utilizing the boundary condition from (1.13) on the cylinder side surface $(r = b_0)$ and (1.6), we find a set of constants δ_n. Indeed, by equating the expression for $\tau_{r\varphi}^\circ$ to zero for $r = b_0$ we obtain that $\delta_n = \lambda_n b_0^{-1}$, where λ_n are roots of the equation $J_2(\lambda_n) = 0$.

Finally, satisfying the boundary conditions for $z = 0$, we will have the following dual series equations to seek the sequence of functions $d_k(t)$

$$l b_0^{-1} d_0(t)\, r + \sum b_0 \lambda_n^{-1} d_n(t)\, J_1(b_0^{-1}\lambda_n r) = \alpha(t)\, r \quad (0 \leqslant r \leqslant a) \tag{1.7}$$
$$b_0^{-1} d_0(t)\, r + \sum d_n(t)\, J_1(b_0^{-1}\lambda_n r) \operatorname{cth}(b_0^{-1}\lambda_n l) = 0 \quad (a \leqslant r \leqslant b)$$

Since $\lambda_n \geqslant \lambda_1 = 3.8317$ and $l b_0^{-1} = \varkappa_0 \gg 1$, then coth $(b_0^{-1}\lambda_n l)$ can be set equal to one with a high degree of accuracy and (1.7) can be investigated in the form /3/

$$u_\varphi(r, 0, t) = \varkappa_0 d_0(t)\, r + \sum b_0 \lambda_n^{-1} d_n(t)\, J_1(b_0^{-1}\lambda_n r) = \alpha(t)\, r \tag{1.8}$$
$$(0 \leqslant r \leqslant a)$$
$$\tau_{\varphi z}^\circ(r, 0, t) = b_0^{-1} d_0(t)\, r + \sum d_n(t)\, J_1(b_0^{-1}\lambda_n r) = 0 \quad (a \leqslant r \leqslant b_0)$$

The dual series Eqs.(1.8) describe the formulated contact problem in the interval $t \in [\tau_0, \tau_1]$, where the time itself occurs in it parametrically. We will now construct the solution of (1.8) below by first obtaining the resolving equations of the problem during continuous growth and after cessation of growth. We merely note that the true stresses are restored according to the $\tau_{r\varphi}^\circ$ and $\tau_{\varphi z}^\circ$ found from the formula

$$\sigma(r, z, t) = G(t)\left[\sigma^\circ(r, z, t) + \int_{\tau_0}^{t} \sigma^\circ(r, z, \tau)\, R_1(t, \tau)\, d\tau\right] \tag{1.9}$$

where $R_1(t, \tau)$ is the resolvent of the kernel $K_1(t, \tau)$.

Let $t \in [\tau_1, \tau_2]$. Then the initial-boundary-value problem for a growing cylinder being twisted by a stamp has the form /1, 2/

$$\frac{\partial \tau_{r\varphi}}{\partial r} + \frac{\partial \tau_{\varphi z}}{\partial z} + \frac{2\tau_{r\varphi}}{r} = 0 \quad (\nabla \cdot \sigma = \mathbf{0}) \tag{1.10}$$

$$z=0,\quad 0\leqslant r\leqslant a:\quad u_\varphi=\alpha(t)\,r;\quad z=0,\quad a\leqslant r\leqslant b(t):\quad \tau_{\varphi z}=0$$
$$r=b(t),\quad 0\leqslant z\leqslant l:\quad \tau_{r\varphi}=0,\quad \tau_{\varphi z}=0\quad (\boldsymbol{\sigma}=\boldsymbol{\sigma}^*=\mathbf{0}),\quad t=\tau^*(r)$$
$$z=l,\quad 0\leqslant r\leqslant b(t):\quad u_\varphi=0$$
$$\dot{\varepsilon}_{r\varphi}=\frac{1}{2}\left(\frac{\partial \dot{u}_\varphi}{\partial r}-\frac{\dot{u}_\varphi}{r}\right),\quad \dot{\varepsilon}_{\varphi z}=\frac{1}{2}\frac{\partial \dot{u}_\varphi}{\partial z}\left(\dot{\boldsymbol{\varepsilon}}=\frac{1}{2}[\nabla\dot{\mathbf{u}}+(\nabla\dot{\mathbf{u}})^T]\right)$$
$$\boldsymbol{\sigma}=2G(t)(\mathbf{I}+\mathbf{F}(\tau_0(r),t))\,\boldsymbol{\varepsilon},\quad \tau_0(r)=\begin{cases}\tau_0, & r\leqslant b_0\\ \tau^*(r), & b_0\leqslant r\leqslant b(t)\end{cases}$$

where the dot denotes differentiation with respect to time, $\boldsymbol{\sigma}^*$ is the stress tensor given on the growth surface that characterizes the clearance of the growing elements ($\boldsymbol{\sigma}^*=\mathbf{0}$ when there is no clearance), $\tau^*(r)$ is the time of attachment of elements with coordinate r to the main cylinder, where $\tau^*(b(\tau))=\tau$.

We write the inverse operator to $(\mathbf{I}+\mathbf{F}(\tau_0(r),t))$, as follows ($h(r-b_0)$ is the Heaviside function):

$$(\mathbf{I}-\mathbf{S}(\tau_0(r),t))\,f(t)=(\mathbf{I}-\mathbf{S}(\tau^\circ(r),t))\,f(t)-[1-h(r-b_0)]\,\mathbf{S}^t(\tau_0,\tau_1)\,f(t) \tag{1.11}$$
$$\mathbf{S}^t(\tau_0,\tau_1)\,f(t)=\int_{\tau_0}^{\tau_1}f(\tau)\,K_1(t,\tau)\,d\tau,\quad \tau^\circ(r)=\tau_1+h(r-b_0)[\tau^*(r)-\tau_1]$$

Acting on the relationships from (1.10) with the operator (1.11) and taking account of the notation $\boldsymbol{\sigma}^\circ=(\mathbf{I}-\mathbf{S}(\tau_0(r),t))\,\boldsymbol{\sigma}G^{-1}$ we will have

$$\frac{\partial\overset{\circ}{\tau}_{r\varphi}}{\partial r}+\frac{\partial\overset{\circ}{\tau}_{\varphi z}}{\partial z}+\frac{2\overset{\circ}{\tau}_{r\varphi}}{r}=0\quad(\nabla\cdot\boldsymbol{\sigma}^\circ=\mathbf{0}) \tag{1.12}$$
$$z=0,\quad 0\leqslant r\leqslant a:\quad u_\varphi=\alpha(t)\,r;\quad z=0,\quad a\leqslant r\leqslant b(t):\quad \tau_{\varphi z}{}^\circ=0$$
$$r=b(t),\quad 0\leqslant z\leqslant l:\quad \tau_{r\varphi}{}^\circ=0,\quad \tau_{\varphi z}{}^\circ=0\quad(\boldsymbol{\sigma}^\circ=\boldsymbol{\sigma}^{\circ*}=\boldsymbol{\sigma}^*G^{-1}=0),$$
$$t=\tau^*(r)$$
$$z=l,\quad 0\leqslant r\leqslant b(t):\quad u_\varphi=0$$
$$\dot{\boldsymbol{\varepsilon}}={}^1/_2[\nabla\dot{\mathbf{u}}+(\nabla\dot{\mathbf{u}})^T],\quad \boldsymbol{\sigma}^\circ=2\boldsymbol{\varepsilon}$$

It is obviously sufficient to show that

$$\frac{\partial}{\partial r}(\mathbf{I}-\mathbf{S}(\tau_0(r),t))\,\tau_{r\varphi}G^{-1}=(\mathbf{I}-\mathbf{S}(\tau_0(r),t))\,\frac{\partial\tau_{r\varphi}}{\partial r}\,G^{-1}$$

for relations (1.12) to be valid.

In fact (see (1.11))

$$\frac{\partial\overset{\circ}{\tau}_{r\varphi}}{\partial r}=(\mathbf{I}-\mathbf{S}(\tau_0(r),t))\,\frac{\partial\tau_{r\varphi}}{\partial r}\,G^{-1}+h(r-b_0)\,\frac{\partial\tau^*(r)}{\partial r}\times$$
$$\tau^*_{r\varphi}G^{-1}(\tau^*(r))\,K_1(t,\tau^*(r))+\delta(r-b_0)\,\mathbf{S}^t(\tau_0,\tau_1)\,\tau_{r\varphi}G^{-1}$$

where the second and third components on the right-hand side of this equality equal zero according to the conditions of the problem: $\tau_{r\varphi}{}^*=0$ and $\tau_{r\varphi}=0$ for $t\in[\tau_0,\tau_1],\ r=b_0,\ 0\leqslant z\leqslant l$ ($\delta(r-b_0)$ is the Dirac delta function).

Now differentiating the equilibrium equation, the governing relationships, and the first, second, and fourth boundary conditions from (1.12) with respect to time and acting on the initial-boundary condition $\boldsymbol{\sigma}^\circ=\boldsymbol{\sigma}^{\circ*}$ with the divergence operator, we obtain the following boundary-value problem

$$\frac{\partial\dot{\overset{\circ}{\tau}}_{r\varphi}}{\partial r}+\frac{\partial\dot{\overset{\circ}{\tau}}_{\varphi z}}{\partial z}+\frac{2\dot{\overset{\circ}{\tau}}_{r\varphi}}{r}=0\quad(\nabla\cdot\boldsymbol{\sigma}^{\circ\cdot}=\mathbf{0}) \tag{1.13}$$
$$z=0,\quad 0\leqslant r\leqslant a:\quad \dot{u}_\varphi=\dot{\alpha}(t)\,r$$
$$z=0,\quad a\leqslant r\leqslant b(t):\quad \tau_{\varphi z}{}^{\circ\cdot}=0$$
$$r=b(t),\quad 0\leqslant z\leqslant l:\quad \tau_{r\varphi}{}^{\circ\cdot}=0,\quad t=\tau^*(r)$$
$$z=l,\quad 0\leqslant r\leqslant b(t):\quad \dot{u}_\varphi=0$$
$$\dot{\boldsymbol{\varepsilon}}={}^1/_2[\nabla\dot{\mathbf{u}}+(\nabla\dot{\mathbf{u}})^T],\quad \boldsymbol{\sigma}^{\circ\cdot}=2\dot{\boldsymbol{\varepsilon}}$$

It is seen that the rate of displacement $u_\varphi^\cdot$ satisfies the equation $Du_\varphi^\cdot = 0$ (see (1.4)) while the expression for $u_\varphi^\cdot$ and the rates of the operator stresses $\tau_{r\varphi}^{\circ\cdot}$ and $\tau_{\varphi z}^{\circ\cdot}$ can be written in the form

$$\begin{gathered} u_\varphi^\cdot(r, z, t) = lb^{-1}(t)\, d_0^\circ(t)\, r\,(1 - zl^{-1}) + \sum \eta_n^{-1}(t)\, d_n^\circ(t)\, J_1[r\eta_n(t)] \times \\ \operatorname{sh}[\eta_n(t)(l - z)]\operatorname{sh}^{-1}[\eta_n(t)\, l] \\ \tau_{\varphi z}^{\circ\cdot}(r, z, t) = -b^{-1}(t)\, d_0^\circ(t)\, r - \sum d_n^\circ(t)\, J_1[r\eta_n(t)]\operatorname{ch}[\eta_n(t)(l - \\ z)]\operatorname{sh}^{-1}[\eta_n(t)\, l] \\ \tau_{r\varphi}^{\circ\cdot}(r, z, t) = -\sum d_n^\circ(t)\, J_2[r\eta_n(t)]\operatorname{sh}[\eta_n(t)(l - z)]\operatorname{sh}^{-1}[\eta_n(t)\, l] \end{gathered} \tag{1.14}$$

Here $d_k^\circ(t)$ $(k = 0, \ldots, \infty)$ and $\eta_n(t)$ $(n = 1, \ldots, \infty)$ are sequences of functions to be determined.

By satisfying the boundary conditions from (1.13), taking into account that $lb^{-1}(t) \gg 1$, we arrive at dual series equations for finding $d_k^\circ(t)$

$$\begin{gathered} u_\varphi^\cdot(r, 0, t) = \varkappa(t)\, d_0^\circ(t)\, r + \sum b(t)\, \lambda_n^{-1} d_n^\circ(t)\, J_1[b^{-1}(t)\, \lambda_n r] = \alpha^\cdot(t)\, r \\ (0 \leqslant r \leqslant a) \\ \tau_{\varphi z}^{\circ\cdot}(r, 0, t) = b^{-1}(t)\, d_0^\circ(t)\, r + \sum d_n^\circ(t)\, J_1[b^{-1}(t)\, \lambda_n r] = 0 \\ (a \leqslant r \leqslant b(t)) \\ \eta_n(t) = \lambda_n b^{-1}(t), \quad \varkappa(t) = lb^{-1}(t), \quad \tau_1 \leqslant t \leqslant \tau_2 \end{gathered} \tag{1.15}$$

If the $d_k^\circ(t)$ are found, meaning $\sigma^{\circ\cdot}$ and $\mathbf{u}^\cdot$ also, the stress tensor σ and the displacement vector $\mathbf{u}$ are established according to the formulas

$$\begin{gathered} \sigma(r, z, t) = G(t)\left\{\frac{\sigma(r, z, \tau_0(r))}{G(\tau_0(r))}\left[1 + \int_{\tau_0(r)}^{t} R_1(t, \tau)\, d\tau\right] + \right. \\ \left. \int_{\tau_0(r)}^{t}\left[\sigma^{\circ\cdot}(r, z, \tau) + \int_{\tau_0(r)}^{\tau}\sigma^{\circ\cdot}(r, z, \zeta)\, d\zeta R_1(t, \tau)\right] d\tau\right\} \\ \mathbf{u}(r, z, t) = \mathbf{u}(r, z, \tau_0(r)) + \int_{\tau_0(r)}^{t}\mathbf{u}^\cdot(r, z, \tau)\, d\tau \end{gathered} \tag{1.16}$$

The boundary-value problem for a growing cylinder has the form (1.10) after the cessation of growth $t \geqslant \tau_2 = \tau^*(b_1)$ where only $b(t) = b_1$ and the usual boundary conditions $\tau_{r\varphi} = 0$ is specified on the cylinder surface. Just as before, it can be reduced to a boundary-value problem in the rates of displacement and operator stresses with a solution in the form (1.14) under the condition $b(t) = b_1$. The resolving dual series equations retain the form (1.15) where $b(t) = b_1, \varkappa(t) = \varkappa_1 = lb_1^{-1}, \eta_n(t) = \eta_n = \lambda_n b_1^{-1}, t \geqslant \tau_2$. After their solution, the stresses $\tau_{r\varphi}, \tau_{\varphi z}$ and the displacement u_φ are determined by using (1.16). It should be noted that the dependence of $\sigma^{\circ\cdot}$ and $\mathbf{u}^\cdot$ on the time t is parametric.

The condition of stamp equilibrium that holds in the whole time interval must be added to the dual series equations obtained

$$M(t) = -2\pi\int_0^a \tau(\rho, t)\, \rho^2\, d\rho, \quad \tau(\rho, t) = \tau_{\varphi z}(\rho, 0, t) \tag{1.17}$$

On the basis of (1.17) the following conditions can also be obtained

$$M^\circ(t) = (\mathbf{I} - \mathrm{S}(\tau_0, t))\, M(t)\, G^{-1}(t) = -2\pi\int_0^a \tau^\circ(\rho, t)\, \rho^2\, d\rho \quad (\tau_0 \leqslant t \leqslant \tau_1) \tag{1.18}$$

$$\begin{gathered} M^{\circ\cdot}(t) = \frac{M^\cdot(t)}{G(t)} + \int_{\tau_0}^{t}\frac{\partial M(\tau)}{\partial\tau}\frac{\partial\omega(t, \tau)}{\partial t}\, d\tau + M(\tau_0)\frac{\partial\omega(t, \tau_0)}{\partial t} = \\ -2\pi\int_0^a \tau^{\circ\cdot}(\rho, t)\, \rho^2\, d\rho \quad (t \geqslant \tau_1) \end{gathered} \tag{1.19}$$

which are more convenient for constructing the solution of the contact problem in a number of cases.

The reasoning presented above is extended to the case $\tau_{\varphi z}{}^* \neq 0$, i.e., a growing cylinder with a certain clearance of the attached elements. For this modification of the problem, only the condition on the growth surface is altered in relations (1.13)

$$r = b\,(t), \quad 0 \leqslant z \leqslant l: \quad \tau_{r\varphi}^{\circ\cdot} = \frac{\partial \tau_{\varphi z}^*}{\partial z} G^{-1} \left| \frac{\partial \tau^*(r)}{\partial r} \right|^{-1}, \quad t = \tau^*(r)$$

Actually, quite natural conditions of no actions on the body surface to which the substance influx will occur, and to the growth surface itself during the growth process /2/ should be satisfied.

2. ***Solution of the dual series equations of the contact problem.*** The resolving dual equations of the problem can be represented in three fundamental time intervals by the single relationships

$$\zeta\varphi_0 x + \sum \lambda_n^{-1} \varphi_n J_1(\lambda_n x) = \psi x \quad (0 \leqslant x \leqslant c) \tag{2.1}$$
$$-p\,(x) = \varphi_0 x + \sum \varphi_n J_1(\lambda_n x) = 0 \quad (c \leqslant x \leqslant 1)$$

where we set $\zeta = \varkappa_0, \varphi_k = d_k\,(t)\ (k = 0, \ldots, \infty), p\,(x) - \tau^\circ(xb_0, t), \psi = \alpha\,(t), c = ab_0{}^{-1}, \ x = rb_0{}^{-1}$, for $t \in [\tau_0, \tau_1]$, we have $\zeta = \varkappa\,(t),\ \varphi_k = d_k{}^\circ(t),\ \psi = \alpha^\cdot(t),\ p\,(x) = \tau^{\circ\cdot}(xb\,(t), t),\ c = ab^{-1}(t),\ x = rb^{-1}(t)$, for $t \in [\tau_1, \tau_2]$ and unlike the preceding $\zeta = \varkappa_1,\ b\,(t) = b_1$ for $t \geqslant \tau_2$.

Let us construct the solution of (2.1) by following /8/ (see /5/ also). Let

$$p\,(x) = \left[\frac{\partial}{\partial x} \int_x^c g\,(\xi)\,(\xi^2 - x^2)^{-1/2}\,d\xi \right] h\,(c - x) \tag{2.2}$$

The series in the second equation of (2.1) is a Dini expansion /9/ of the function $-p\,(x)$, whose coefficients $\varphi_k\ (k = 0, \ldots, \infty)$ are given by the formulas

$$\varphi_0 = -4 \int_0^1 x^2 p\,(x)\,dx = 8 \int_0^c \xi g\,(\xi)\,d\xi \tag{2.3}$$
$$\varphi_n = -2J_1^{-2}(\lambda_n) \int_0^1 x p\,(x)\, J_1(\lambda_n x)\,dx = 2J_1^{-2}(\lambda_n) \int_0^c g\,(\xi) \sin(\lambda_n \xi)\,d\xi$$
$$(n = 1, \ldots, \infty)$$

when (2.2) is taken into account.

Substituting (2.3) into the first equation of (2.1) and using the technique from /8, 10-13/, we obtain a Fredholm integral equation of the second kind to determine the function $g\,(x)$

$$g\,(x) + \int_0^c g\,(\xi)\, k\,(x, \xi)\,d\xi = \frac{4\psi x}{\pi} \quad (1 \leqslant x \leqslant c) \tag{2.4}$$
$$k\,(x, \xi) = \frac{16}{\pi}(1 - 2\zeta)\, x\xi + \frac{4}{\pi^2} \int_0^\infty \frac{K_2(y)}{I_2(y)} [8x\xi I_2(y) - \operatorname{sh} xy \operatorname{sh} \xi y]\,dy$$

where $K_\nu(y), I_\nu(y)$ are Bessel functions of imaginary argument of order ν.

The solution of (2.4) obviously yields the complete solution of the contact problem in question also. It can be found numerically /3/ or by using iteration methods /5, 14/. We consider here one method, proposed in /3/, for constructing the approximate solution of (2.4). We note that for $\zeta \geqslant 10$ the deviation of the approximate from the numerical solution does not exceed 8.5% for $c = 0.7$, 7% for $c = 0.6$ and 1% for $c \leqslant 0.5$.

We will use the fact that the quantity ζ is fairly large and we will limit ourselves to the first term in the expression for the kernel $k\,(x, \xi)$ (see (2.4))

$$g\,(x) + \frac{16}{\pi}(1 - 2\zeta)\, x \int_0^c g\,(\xi)\,\xi\,d\xi = \frac{4\psi x}{\pi} \quad (1 \leqslant x \leqslant c) \tag{2.5}$$

Then substituting $g(x) = Ax$ into (2.5) and determining A, we will have by virtue of (2.2)

$$p(x) = -4\psi\,[\pi + 16\,(2\zeta - 1)\,c^3/3]^{-1}\,x\,(c^2 - x^2)^{-1/2}\ (1 \leqslant x \leqslant c) \tag{2.6}$$

The dependences of the operator contact stresses and their rates on the angle of stamp rotation can be written on the basis of (2.6) in the form $(0 \leqslant r \leqslant a)$

$$\tau^\circ(r, t) = \alpha(t)\,W(r, b_0)\quad (\tau_0 \leqslant t \leqslant \tau_1) \tag{2.7}$$

$$\tau^{\circ\cdot}(r, t) = \alpha^\cdot(t)\,W(r, b(t))\quad (\tau_1 \leqslant t \leqslant \tau_2) \tag{2.8}$$

$$\tau^{\circ\cdot}(r, t) = \alpha^\cdot(t)\,W(r, b_1)\quad (t \geqslant \tau_2) \tag{2.9}$$

$$W(r, \xi) = -4/[\pi + 16\,(2l\xi^{-1} - 1)\,a^3\xi^{-3}/3],\quad r\,(d^2 - r^2)^{-1/2}$$

For a given angle of stamp rotation $\tau^\circ(r, t), \tau^{\circ\cdot}(r, t)$ are found at once from (2.7)-(2.9) and by using the relationships described earlier the contact stresses $\tau(r, t)$ are restored. The moment acting on the stamp is calculated from (1.17). We note that for $\alpha(t) = \text{const}$ the mutual influence of the initial cylinder and its newly forming unstressed part does not appear.

On the basis of (1.9), (1.18) and (2.7) we will have for a given torque $M(t)$

$$\tau(r, t) = 3M(t)\,(4\pi a^3)^{-1} r\,(a^2 - r^2)^{-1/2} \tag{2.10}$$

$$\alpha(t) = B(b_0)\,(\mathbf{I} - \mathbf{S}(\tau_0, t))\,M(t)/G(t)\quad (\tau_0 \leqslant t \leqslant \tau_1)$$

$$B(\xi) = 3/(16a^3) + (2l - \xi)/(\pi\xi^4)$$

Using (1.16), (1.19), (2.8) and (2.9), we finally obtain the relationship (2.10) for the contact stresses for $t \geqslant \tau_1$, and the following expressions for the angle of rotation

$$\alpha^\cdot(t) = M^{\circ\cdot}(t)\,B(b(t)),\quad \alpha(t) = \alpha(\tau_1) + \int_{\tau_1}^{t} \alpha^\cdot(\tau)\,d\tau\quad (\tau_1 \leqslant t \leqslant \tau_2)$$

$$\alpha^\cdot(t) = M^{\circ\cdot}(t)\,B(b_1),\quad \alpha(t) = \alpha(\tau_2) + \int_{\tau_2}^{t} \alpha^\cdot(\tau)\,d\tau\quad (t \geqslant \tau_2)$$

It turns out that the growth of a cylinder during torsion of a stamp by a moment of forces has a slight influence on the contact stress distribution if the stamp and cylinder radii are not very close (specific ratios are given above). However, a substantial dependence of the angle of stamp rotation on the time from when the cylinder starts to grow and on the growth rate appears in this same case.

3. ***Numerical example.*** We examine the contact problem in question by considering the cylinder to be fabricated from concrete with a modulus of elastically instantaneous shear strain $G(t) = G = \text{const}$ and a measure of creep under shear in the form /15, 16/

$$\omega(t, \tau) = (D_0 + Fe^{-\beta\tau})\,(1 - e^{-\gamma(t-\tau)})$$

We will make a change of variables according to the formulas

$$r^* = ra^{-1},\ \rho^* = \rho a^{-1},\ t^* = t\tau_0^{-1},\ \tau^* = \tau\tau_0^{-1},\ \tau^*(r^*, t^*) = \tau(r, t)\,G^{-1}$$

$$\tau_1^* = \tau_1\tau_0^{-1},\ \tau_2^* = \tau_2\tau_0^{-1},\ M^*(t^*) = M(t)\,G^{-1}a^{-3},\ \alpha^*(t^*) = \alpha(t)$$

$$\beta^* = \beta\tau_0,\ \gamma^* = \gamma\tau_0,\ b_0^* = b_0 a^{-1},\ b_1^* = b_1 a^{-1}$$

$$b^*(t^*) = b(t)\,a^{-1},\ l^* = la^{-1},\ D_0^* = D_0 G,\ F^* = FG$$

and omitting the asterisk in the notation, we give the following values of the functions and parameters:

$$b_0 = 1/0.7,\ l = 20/0.7,\ b(t) = b_0\,(t + \tau_2 - 2\tau_1)\,(\tau_2 - \tau_1)^{-1}$$

$$b_1 = 2b_0,\ M(t) = 1,\ D_0 = 0.251,\ F = 1.818$$

$$\beta = 0.31,\ \gamma = 0.6,\ \tau_0 = 10\ \text{days}$$

It is seen that during the time of growth the cylinder radius doubles. The growth rate is constant and is determined only by the times of the beginning and cessation of growth. The torque acting on the stamp does not change with time. Moreover, the ratio of the cylinder length to its radius is greater than or equal to 10 during the extent of the whole process, while the ratio of the stamp and cylinder radii does not exceed 0.7, i.e., formulas of the approximate solution can be utilized.

As regards the contact stress distribution, it is sufficient to refer to (2.10) to note that it (the distribution) is practically independent of the properties of the material and for a constant torque can be considered to be invariant $(\tau(r,t) = 3r/[4\pi(1-r^2)^{1/2}])$ with a sufficient degree of accuracy.

The behaviour of the angle of stamp rotation as a function of the fundamental characteristics of the process of piecewise-continuous cylinder growth requires a more detailed analysis.

The three lower curves in Fig.2 show the change in the angle of rotation α in a time t for a cylinder whose growth starts simultaneously with the application of the torque $(\tau_1 = 1)$ for different growth rates $b^{\cdot}(t)$: $b^{\cdot}(t) = b_0/9$ $(\tau_2 = 10)$ is the solid line $b^{\cdot}(t) = b_0/3$ $(\tau_2 = 4)$ is the dash-dot line, and $b^{\cdot}(t) = b_0$ $(\tau_2 = 2)$ is the dashed line. The times of the halts in growth are marked by the vertical solid lines.

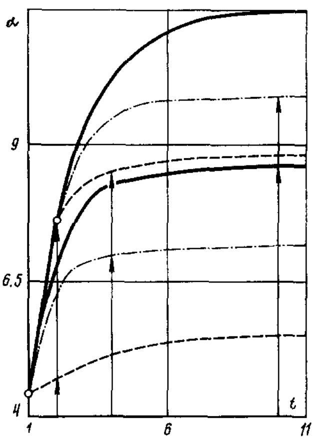

Fig.2

The three upper curves in Fig.2 correspond to dependences of the angle of stamp rotation on the time for different growth rates $b^{\cdot}(t)$ for a cylinder loaded at the time 1 and starting to grow at the time $\tau_1 = 2$: $b^{\cdot}(t) = b_0/8$ $(\tau_2 = 10)$ the dash-dot line, and $b^{\cdot}(t) = b_0/2$ $(\tau_2 = 4)$ the dashed line. For comparison, the change in the angle of stamp rotation that twists a cylinder of fixed radius b_0 is shown by the solid line. The sections of the curves located between the vertical solid lines characterize the behaviour of the angle of stamp rotation in intervals of continuous cylinder growth.

The graphs demonstrate the essential dependence of the angle of rotation $\alpha(t)$ on the growth rate and the time of the beginning of growth. Thus the limit value of the increment in the angle of stamp rotation $\Delta(\infty)$ $(\Delta(t) = \alpha(t) - \alpha(\tau_0))$ during slow cylinder growth can exceed the same value for rapid growth by a factor of 2.7. The characteristic time, starting with which the influence of the process of piecewise-continuous growth of the contact interaction characteristics can be neglected exists for a constant torque. In the same case a strong dependence of the limit value of the angle of stamp rotation on the time interval between times of the beginning of loading and the beginning of growth appears.

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AN INEQUALITY IN THE THEORY OF A SEMILINEAR ELASTIC BODY*

V.A. MISYURA

An inequality for a geometrically non-linear problems is obtained as an analogue of the Prager-Synge identity in linear elasticity theory on the basis of a representation of the elastic energy density of a semilinear elastic body.

The convexity of the potential energy functional in geometrically linear problems of elasticity theory enabled a dual variational problem, the Castigliano principle, to be formulated. The fact that the lower bound of the direct functional I is associated with the lower bound of the dual by the relationship

$$\inf I = -\inf J = \sup(-J) \tag{0.1}$$

turns out to be remarkable here.

The potential energy functional I is examined in a set of kinematically allowable displacement fields $\mathbf{w}$, the dual J in a set of statically allowable stress fields σ. The property (0.1) of the dual problem enables the minimum value of the direct functional $I(\mathbf{w})$ to be estimated as accurately as desired from below. But this would at once yield /1/ an estimate of the approximation $\mathbf{w}$ minimizing the element $\mathbf{w}^\circ$ in the norm L_2

$$\|\mathbf{w}-\mathbf{w}^\circ\|_{L_2(V_0)} \leqslant C\,(I(\mathbf{w})-d)^{1/2} \tag{0.2}$$

where $d \leqslant I(\mathbf{w}^\circ)$ is the lower limit of the minimum value of the functional I, C is a constant, and V_0 is the domain occupied by the elastic body in the undeformed state.

The estimate (0.2) can be reduced to the form /2/

$$\|\bar{\sigma}-\sigma^\circ\|_{L_2(V_0)} \leqslant C\,\|\bar{\sigma}-\sigma'\|_{L_2(V_0)} \tag{0.3}$$

Here $\bar{\sigma}$ is a statically allowable stress field, σ' is the kinematically allowable stress, and σ° is the true state of stress of the elastic body.

The natural desire to extend these results to the case of geometrically non-linear problems of elasticity theory encounters a number of difficulties in principle. The first is associated with the fact that the potential energy functional in geometrically non-linear problems is not convex. In substance, this excludes the possiblity of constructing a dual functional for which condition (0.1) would be satisfied. It is thereby impossible to compute the lower bound of the potential energy functional as exactly as desired. The second difficulty is that the relationship (0.2) is not valid in geometrically non-linear problems. And even in the case when the dual problem /4/ is constructed formally according to standard procedure /3/ and a lower bound of the minimum value of the direct functional is obtained, the connection between this estimate and the error of the approximate solution is not clear.

An attempt is made below to obtain an inequality of the type (0.3) for a semilinear

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